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Surfaces of Demoulin: differential geometry, Bäcklund transformation and integrability

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Abstract

The surfaces of Demoulin constitute an important subclass of surfaces in projective differential geometry which arise in many seemingly unrelated geometric constructions. Analytically, they are described by a two-component system which coincides with the $D_3^{(2)}$ Toda lattice. We review some of the most important geometric properties of the Demoulin surfaces and construct a Bäcklund transformation which may be specialized to the well-known Bäcklund transformation for the Tzitzeica equation governing affine spheres in affine geometry. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In recent years, it has become evident that classical differential geometry constitutes a remarkable source of integrable systems. The best-known examples include constant Gaussian curvature, constant mean curvature and Bianchi surfaces in Euclidean geometry [3], affine spheres in affine geometry [24] and Willmore [28] and isothermic surfaces [5]

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in conformal geometry. Others are contained in the theory of orthogonal and conjugate coordinate systems [7,10].

In the present paper, we focus on the less familiar case of surfaces in projective differential geometry which does not seem to have attracted attention in the context of integrability. As a first important example, we here analyse the so-called Demoulin surfaces $M^2 \in P^3$ which are described analytically by the system

$$(\ln p)_{xy} = pq + \frac{1}{p}, \qquad (\ln q)_{xy} = pq + \frac{1}{q}. \tag{1}$$

The latter turns out to represent the $D_3^{(2)}$ Toda lattice. The position vector $\boldsymbol{r} = (r^0 : r^1 : r^2 : r^3)$ of the surface M^2 satisfies the linear system

$$\boldsymbol{r}_{xx} = p\boldsymbol{r}_y + \frac{1}{2}\left(\frac{q_{xx}}{q} - \frac{1}{2}\left(\frac{q_x}{q}\right)^2 - p_y\right)\boldsymbol{r},$$
$$\boldsymbol{r}_{yy} = q\boldsymbol{r}_x + \frac{1}{2}\left(\frac{p_{yy}}{p} - \frac{1}{2}\left(\frac{p_y}{p}\right)^2 - q_x\right)\boldsymbol{r},$$

which is compatible if the functions p, q obey (1). The surfaces of Demoulin play a central role in projective differential geometry and arise in a number of natural geometric constructions, some of which will be discussed here. For instance, they appear in the theory of envelopes of Lie quadrics associated with the surface M^2 and generate Laplace sequences of period 6 via the Plücker embedding in P^5.

After briefly recalling in Sections 2–4 the general approach to projective differential geometry of surfaces, we derive in Sections 5 and 6 the equations of motion for the so-called Wilczynski tetrahedral. These give rise to the 4×4 spectral problem

$$\phi_x = \begin{pmatrix} \frac{1}{2}\frac{q_x}{q} & \lambda & 0 & 0 \\ 0 & -\frac{1}{2}\frac{q_x}{q} & \lambda p & 0 \\ -\frac{\lambda}{2p} & 0 & \frac{1}{2}\frac{q_x}{q} & \lambda \\ 0 & -\frac{\lambda}{2p} & 0 & -\frac{1}{2}\frac{q_x}{q} \end{pmatrix}\phi,$$
$$\phi_y = \begin{pmatrix} \frac{1}{2}\frac{p_y}{p} & 0 & \frac{1}{\lambda} & 0 \\ -\frac{1}{2q\lambda} & \frac{1}{2}\frac{p_y}{p} & 0 & \frac{1}{\lambda} \\ 0 & \frac{q}{\lambda} & -\frac{1}{2}\frac{p_y}{p} & 0 \\ 0 & 0 & -\frac{1}{2q\lambda} & -\frac{1}{2}\frac{p_y}{p} \end{pmatrix}\phi \tag{2}$$

of system (1). In Section 7, we introduce a different second-order spectral problem

$$
\begin{aligned}
\varphi_{xx} &= -\frac{q_x}{q}\varphi_x + \lambda p\psi_y, & \psi_{xx} &= -\frac{p_x}{p}\psi_x + \lambda q\varphi_y,\\
\varphi_{xy} &= -\frac{1}{q}\varphi, & \psi_{xy} &= -\frac{1}{p}\psi,\\
\varphi_{yy} &= -\frac{q_y}{q}\varphi_y + \frac{1}{\lambda}p\psi_x, & \psi_{yy} &= -\frac{p_y}{p}\psi_y + \frac{1}{\lambda}q\varphi_x,
\end{aligned}
\tag{3}
$$

which is related to (2) through the Plücker correspondence discussed in Section 8. Geometrically, the spectral problem (3) makes manifest the fact that Demoulin surfaces generate Laplace sequences of period 6 via the Plücker embedding in P^5. Under the reduction $p = q$, $\varphi = \psi$, the linear system (3) specializes to the linear representation of the Tzitzeica equation

$$
(\ln h)_{xy} = h - \frac{1}{h^2}, \qquad hp = -1, \tag{4}
$$

governing affine spheres in affine differential geometry. In Sections 9 and 10, we derive a Bäcklund transformation for system (1) which is, in geometric terms, generated by a W-congruence.

2. Surfaces in projective geometry

Based on [25–27], let us briefly recall the standard way of defining surfaces M^2 in projective space P^3 in terms of solutions of a linear system

$$
\boldsymbol{r}_{xx} = p\boldsymbol{r}_y + \pi\boldsymbol{r}, \quad \boldsymbol{r}_{yy} = q\boldsymbol{r}_x + \chi\boldsymbol{r}, \tag{5}
$$

where p, π, q, χ are functions of x and y. If we cross-differentiate (5) and assume $\boldsymbol{r}, \boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{r}_{xy}$ to be independent, we arrive at the compatibility conditions

$$
\begin{aligned}
&\pi_{yy} + 2p_y\chi + p\chi_y = \chi_{xx} + 2q_x\pi + q\pi_x,\\
&\qquad (q_x + 2\chi)_x = 2qp_y + pq_y, \qquad (p_y + 2\pi)_y = 2pq_x + qp_x.
\end{aligned}
$$

These may be cast into the form [16, p. 120]

$$
p_{yyy} - 2p_yW - pW_y = q_{xxx} - 2q_xV - qV_x, \tag{6a}
$$

$$
W_x = 2qp_y + pq_y, \tag{6b}
$$

$$
V_y = 2pq_x + qp_x, \tag{6c}
$$

by introducing $W = q_x + 2\chi$, $V = p_y + 2\pi$. For any fixed solution p, q, V, W of (6a)–(6c) the linear system (5) is compatible and possesses exactly four linearly independent solutions $\boldsymbol{r} = (r^0, r^1, r^2, r^3)$ which can be regarded as homogeneous coordinates of a surface in projective space. For our purposes, one may think of M^2 as a surface in a three-dimensional Euclidean space with position vector $\boldsymbol{R} = (r^1/r^0, r^2/r^0, r^3/r^0)$. If we choose any other four solutions $\tilde{\boldsymbol{r}} = (\tilde{r}^0, \tilde{r}^1, \tilde{r}^2, \tilde{r}^3)$ of the same system (5) then the corresponding surface $\tilde{M}^2$ with position vector $\tilde{\boldsymbol{R}} = (\tilde{r}^1/\tilde{r}^0, \tilde{r}^2/\tilde{r}^0, \tilde{r}^3/\tilde{r}^0)$ constitutes a projective transform of M^2 so that any fixed solution p, q, V, W of equations (6a)–(6c) defines a surface M^2 uniquely up to projective equivalence. Moreover, a simple calculation yields

$$\boldsymbol{R}_{xx} = p\boldsymbol{R}_y + a\boldsymbol{R}_x, \qquad \boldsymbol{R}_{yy} = q\boldsymbol{R}_x + b\boldsymbol{R}_y$$

($a = -2r^0_x/r^0$, $b = -2r^0_y/r^0$) which implies that x, y are asymptotic coordinates of the surface M^2. In what follows, we assume that our surfaces are hyperbolic and the corresponding asymptotic coordinates x, y are real.[2] Since Eqs. (6a)–(6c) specify a surface uniquely up to projective equivalence, they can be viewed as the 'Gauss–Codazzi' equations in projective geometry.

Remark. *For any solution* $\boldsymbol{r} = (r^0, r^1, r^2, r^3)$ *of system* (5) *the determinant* $\det(\boldsymbol{r}, \boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{r}_{xy})$ *of the* 4×4 *matrix formed by the vectors* $\boldsymbol{r}, \boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{r}_{xy}$ *is independent of* x, y *and may be normalized to* 1.

Different types of surfaces can be defined by imposing additional constraints on p, q, V, W so that, in a sense, projective differential geometry is the theory of (integrable) reductions of the underdetermined system (6a)–(6c).

Example 1. Isothermally asymptotic surfaces (see e.g. [4, p. 317]) are specified by the condition $p = q$, in which case Eqs. (6a)–(6c) assume the form of the stationary modified Veselov–Novikov (mVN) equation

$$p_{yyy} - 2p_yW - pW_y = p_{xxx} - 2p_xV - pV_x,$$
$$W_x = \frac{3}{2}(p^2)_y, \qquad V_y = \frac{3}{2}(p^2)_x.$$

This fact has been pointed out in [11]. Therein, it has also been shown that a similar class of surfaces (the so-called diagonally cyclidic surfaces) arise in Lie sphere geometry. These are described by a different real form of the stationary mVN equation.

Particular reductions of system (6a)–(6c) corresponding to important classes of surfaces were investigated in the classical context of projective differential geometry. Some of them, namely projectively minimal, Godeaux–Rozet and Demoulin surfaces, will be discussed below in the context of modern integrability.

3. Invariants of surfaces in projective geometry

Even though the coefficients p, q, V, W define a surface M^2 uniquely up to projective equivalence via

$$\boldsymbol{r}_{xx} = p\boldsymbol{r}_y + \tfrac{1}{2}(V - p_y)\boldsymbol{r}, \qquad \boldsymbol{r}_{yy} = q\boldsymbol{r}_x + \tfrac{1}{2}(W - q_x)\boldsymbol{r}, \tag{7}$$

[2] The elliptic case is dealt with in an analogous manner by regarding x, y as complex conjugates.

it is not entirely correct to regard p, q, V, W as projective invariants. Indeed, the asymptotic coordinates x, y are only defined up to an arbitrary reparametrization of the form

$$x^* = f(x), \quad y^* = g(y), \tag{8}$$

which induces a scaling of the surface vector according to

$$\boldsymbol{r}^* = \sqrt{f'(x)g'(y)}\,\boldsymbol{r}. \tag{9}$$

Thus [4, p. 1], the form of Eq. (7) is preserved by the above transformation with the new coefficients p^*, q^*, V^*, W^* given by

$$\begin{aligned} p^* &= pg'/(f')^2, \quad V^*(f')^2 = V + S(f),\\ q^* &= qf'/(g')^2, \quad W^*(g')^2 = W + S(g), \end{aligned} \tag{10}$$

where $S(\cdot)$ is the usual Schwarzian derivative, that is

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

The transformation formulae (10) imply that the symmetric 2-form

$$pq\ \mathrm{d}x\,\mathrm{d}y$$

and the conformal class of the cubic form

$$p\,\mathrm{d}x^3 + q\,\mathrm{d}y^3$$

are absolute projective invariants. They are known as the projective metric and the Darboux cubic form, respectively, and play an important role in projective differential geometry since, in particular, they define a 'generic' surface uniquely up to projective equivalence.

Remark. *The transformation properties of V and W suggest their interpretation as projective connections (along the x- and y-asymptotic lines, respectively). Let us consider, for instance, the ordinary differential equation*

$$\frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2 + V = 0 \tag{11}$$

in the variable x along the asymptotic line $y = y^0 =$ const. Eq. (11) is defined in an invariant way in view of the transformation properties of V. Since any solution of (11) can be represented in the form

$$\varphi = u^1/u^2, \tag{12}$$

where u^1, u^2 are two arbitrary solutions of the linear equation

$$u_{xx} = \tfrac{1}{2}Vu,$$

the function φ defines a projective structure along the asymptotic line $y = y^0 =$ const. Note that φ is only determined up to linear fractional transformations $\varphi \to (a\varphi + b)/(c\varphi + d)$. Thus, V and W define canonical projective structures along the respective asymptotic lines.

4. The Wilczynski moving tetrahedral

Using (8)–(10), one can easily verify that the four points

$$\begin{aligned} &\boldsymbol{r}, \quad \boldsymbol{r}_1 = \boldsymbol{r}_x - \frac{1}{2}\frac{q_x}{q}\boldsymbol{r}, \quad \boldsymbol{r}_2 = \boldsymbol{r}_y - \frac{1}{2}\frac{p_y}{p}\boldsymbol{r}, \\ &\boldsymbol{\eta} = \boldsymbol{r}_{xy} - \frac{1}{2}\frac{q_x}{q}\boldsymbol{r}_y - \frac{1}{2}\frac{p_y}{p}\boldsymbol{r}_x + \left(\frac{1}{4}\frac{p_y q_x}{pq} - \frac{1}{2}pq\right)\boldsymbol{r} \end{aligned} \tag{13}$$

are defined in an invariant way, that is under the transformation formulae (8)–(10) they acquire a nonzero multiple which does not change them as points in projective space P^3. These points form the vertices of the so-called Wilczynski moving tetrahedral [4]. Since the lines $(\boldsymbol{r}, \boldsymbol{r}_1)$ and $(\boldsymbol{r}, \boldsymbol{r}_2)$ are tangential to the x- and y-asymptotic lines, respectively, the three points $\boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_2$ span the tangent plane of the surface M^2 at $\boldsymbol{r}$. The line $(\boldsymbol{r}_1, \boldsymbol{r}_2)$ lying in the tangent plane is known as the directrix of Wilczynski of the second kind. The line $(\boldsymbol{r}, \boldsymbol{\eta})$ is transversal to M^2 and is known as the directrix of Wilczynski of the first kind. It plays the role of a projective 'normal'. We stress that in projective differential geometry there exists no unique choice of an invariant normal. This is in contrast with Euclidean and affine geometries in which the normal is canonically defined. Some of the best-known and most-investigated normals are those of Wilczynski, Fubini, Green, Darboux, Bompiani and Sullivan [4, p. 35] with the directrix of Wilczynski being the most commonly used. It is known that the directrix of Wilczynski intersects the tangent Lie quadric (cf. Section 5) of the surface M^2 at exactly two points $\boldsymbol{r}$ and $\boldsymbol{\eta}$ so that both points lie on the Lie quadric and are canonically defined. The Wilczynski tetrahedral proves to be the most convenient tool in projective differential geometry.

5. Projectively minimal, Godeaux–Rozet and Demoulin surfaces

The metric $pq\,\mathrm{d}x\,\mathrm{d}y$ is invariant under projective transformations and hence gives rise to the projective area functional

$$\int\!\!\int pq\,\mathrm{d}x\,\mathrm{d}y. \tag{14}$$

Its extrema are known as projectively minimal surfaces. The Euler–Lagrange equations for the functional (14) adopt the form [4, p. 319]

$$p_{yyy} - 2p_yW - pW_y = 0, \tag{15a}$$

$$q_{xxx} - 2q_xV - qV_x = 0, \tag{15b}$$

$$W_x = 2qp_y + pq_y, \tag{15c}$$

$$V_y = 2pq_x + qp_x, \tag{15d}$$

and may also be obtained by equating to zero both sides of Eq. (6a). Multiplication of (15a) by p and (15b) by q and subsequent integration results in

$$W = \frac{p_{yy}}{p} - \frac{1}{2}\left(\frac{p_y}{p}\right)^2 + \frac{\varphi(x)}{p^2}, \tag{16a}$$

$$V = \frac{q_{xx}}{q} - \frac{1}{2}\left(\frac{q_x}{q}\right)^2 + \frac{\psi(y)}{q^2}, \tag{16b}$$

where φ and ψ are arbitrary functions of integration. Modulo interchanging φ and ψ, there are three cases to distinguish:

Case 1 (*General case*). Both $\varphi(x)$ and $\psi(y)$ are nonzero. In this case, we can always normalize $\varphi(x), \psi(y)$ to ± 1 by means of the transformations (10). Let us assume, for instance, that $\varphi(x) = \psi(y) = 1$. Insertion of (16a) with $\varphi = 1$ into (15c) and (16b) with $\psi = 1$ into (15d) results in the system

$$[p(\ln p)_{xy} - p^2 q]_y = 2\frac{p_x}{p^2}, \qquad [q(\ln q)_{xy} - q^2 p]_x = 2\frac{q_y}{q^2},$$

or, equivalently,

$$\begin{aligned} (\ln p)_{xy} &= pq + \frac{A}{p}, \quad & A_y &= 2\frac{p_x}{p^2}, \\ (\ln q)_{xy} &= pq + \frac{B}{q}, \quad & B_x &= 2\frac{q_y}{q^2}. \end{aligned} \tag{17}$$

Case 2 (*Surfaces of Godeaux–Rozet* [4, p. 318]). In this case, $\varphi = 0$, while ψ is nonzero and may be normalized to ± 1. Here, we assume that $\psi = 1$. On inserting (16a) with $\varphi = 0$ into (15c) and (16b) with $\psi = 1$ into (15d) we obtain

$$[p(\ln p)_{xy} - p^2 q]_y = 0, \qquad [q(\ln q)_{xy} - q^2 p]_x = 2\frac{q_y}{q^2}.$$

Integration of the first equation produces $(\ln p)_{xy} = pq + s(x)/p$. Hence, if $s(x)$ is nonzero, it may be reduced to 1 by means of (10) so that the resulting equations take the form

$$(\ln p)_{xy} = pq + \frac{1}{p}, \quad (\ln q)_{xy} = pq + \frac{B}{q}, \quad B_x = 2\frac{q_y}{q^2}. \tag{18}$$

Case 3 (*Surfaces of Demoulin*). In this case, both φ and ψ are zero and insertion of (16a) and (16b) with $\varphi = \psi = 0$ into (15c) and (15d) yields

$$[p(\ln p)_{xy} - p^2 q]_y = 0, \qquad [q(\ln q)_{xy} - q^2 p]_x = 0,$$

so that

$$(\ln p)_{xy} = pq + \frac{s(x)}{p}, \qquad (\ln q)_{xy} = pq + \frac{t(y)}{q}.$$

Once again, the analysis falls into three subcases depending on whether s, t are zero or not. Here, we consider only the generic situation $s \neq 0, t \neq 0$ in which case s and t may be normalized to 1 and the resulting equations assume the form (1)

$$(\ln p)_{xy} = pq + \frac{1}{p}, \qquad (\ln q)_{xy} = pq + \frac{1}{q}.$$

In this form, the equations governing Demoulin surfaces have been set down in [12, p. 51]. The same system has been presented in [18] as a reduction of the two-dimensional Toda lattice (cf. Section 8).

On use of the symmetry $p \to \lambda p$, $q \to q/\lambda$ of Eqs. (15a)–(15d), a parameter may be inserted into the linear equations (7) for projectively minimal surfaces. They become

$$\boldsymbol{r}_{xx} = \lambda p \boldsymbol{r}_y + \frac{1}{2}(V - \lambda p_y)\boldsymbol{r}, \qquad \boldsymbol{r}_{yy} = \frac{1}{\lambda} q \boldsymbol{r}_x + \frac{1}{2}(W - \frac{1}{\lambda} q_x)\boldsymbol{r}.$$

Remarkably, this observation was exploited by Demoulin [8] to establish in a purely geometric manner the existence of Bäcklund transformations for Godeaux–Rozet and Demoulin surfaces and associated permutability theorems. Apparently, Demoulin did not formulate his results in terms of analytic expressions. In Section 9, a Toda lattice connection is used to derive explicitly a Bäcklund transformation for Demoulin surfaces.

Remark. *The specialization $p = q$ reduces* (1) *to the Tzitzeica equation* (4) *which governs affine spheres in affine differential geometry* [24].[3] *Geometrically this means that affine spheres lie in the intersection of two different integrable classes of projective surfaces, namely isothermally asymptotic and projectively minimal surfaces.*

Projectively minimal, Godeaux–Rozet and Demoulin surfaces also arise in the theory of envelopes of Lie quadrics associated with the surface M^2. For brevity, we only recall the necessary definitions. The details can be found in [4]. Thus, let us consider a point p^0 on the surface M^2 and the x-asymptotic line passing through p^0. Let us take three additional points p^i, $i = 1, 2, 3$ on this asymptotic line close to p^0 and draw three y-asymptotic lines γ^i passing through p^i. The three straight lines which are tangential to γ^i and pass through the points p^i uniquely define a quadric $\boldsymbol{Q}$ containing them as rectilinear generators. As p^i tend to p^0, the quadric $\boldsymbol{Q}$ tends to a limiting quadric, the so-called Lie quadric of the surface M^2 at the point p^0. Even though this construction depends on the initial choice of either the x- or the y-asymptotic line through p^0, the resulting quadric $\boldsymbol{Q}$ is independent of that choice. Thus, we arrive at a two-parameter family of quadrics associated with the surface M^2. In terms of the Wilczynski tetrahedral, the parametric equation for $\boldsymbol{Q}$ is of the form [4, p. 311]

$$\boldsymbol{Q} = \boldsymbol{\eta} + \mu \boldsymbol{r}_1 + \nu \boldsymbol{r}_2 + \mu\nu \boldsymbol{r},$$

where μ, ν are parameters.

---

[3] Affine spheres may be regarded as the analogues of spheres in affine differential geometry [2].

Now, in the neighbourhood of a generic point p^0 on M^2, the envelopes of the family of Lie quadrics consist of the surface M^2 itself and four, in general, distinct sheets. Surfaces of Godeaux–Rozet are characterized by the degenerate case of two distinct sheets while Demoulin surfaces are present if all four sheets coincide. Another interesting property of Demoulin surfaces is discussed below. Surfaces of Godeaux–Rozet and Demoulin have been investigated extensively in [8,13,22].

6. Surfaces of Demoulin: the 4×4 linear problem

In the case of Demoulin surfaces, p and q satisfy Eq. (1) while V and W read

$$V = \frac{q_{xx}}{q} - \frac{1}{2}\left(\frac{q_x}{q}\right)^2, \qquad W = \frac{p_{yy}}{p} - \frac{1}{2}\left(\frac{p_y}{p}\right)^2,$$

so that Eq. (7) for the position vector $\boldsymbol{r}$ assume the form

$$\begin{aligned} \boldsymbol{r}_{xx} &= p\boldsymbol{r}_y + \frac{1}{2}\left(\frac{q_{xx}}{q} - \frac{1}{2}\left(\frac{q_x}{q}\right)^2 - p_y\right)\boldsymbol{r}, \\ \boldsymbol{r}_{yy} &= q\boldsymbol{r}_x + \frac{1}{2}\left(\frac{p_{yy}}{p} - \frac{1}{2}\left(\frac{p_y}{p}\right)^2 - q_x\right)\boldsymbol{r}. \end{aligned} \tag{19}$$

In terms of the Wilczynski tetrahedral $\boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{\eta}$ (cf. Section 4), the linear system (19) is of first order in the derivatives of p and q. Indeed, using (1) and (19), we easily derive for $\boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{\eta}$ the linear equations

$$\begin{aligned} \begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{\eta} \end{pmatrix}_x &= \begin{pmatrix} \frac{1}{2}\frac{q_x}{q} & 1 & 0 & 0 \\ 0 & -\frac{1}{2}\frac{q_x}{q} & p & 0 \\ -\frac{1}{2p} & 0 & \frac{1}{2}\frac{q_x}{q} & 1 \\ 0 & -\frac{1}{2p} & 0 & -\frac{1}{2}\frac{q_x}{q} \end{pmatrix} \begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{\eta} \end{pmatrix}, \\ \begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{\eta} \end{pmatrix}_y &= \begin{pmatrix} \frac{1}{2}\frac{p_y}{p} & 0 & 1 & 0 \\ -\frac{1}{2q} & \frac{1}{2}\frac{p_y}{p} & 0 & 1 \\ 0 & q & -\frac{1}{2}\frac{p_y}{p} & 0 \\ 0 & 0 & -\frac{1}{2q} & -\frac{1}{2}\frac{p_y}{p} \end{pmatrix} \begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{\eta} \end{pmatrix}, \end{aligned} \tag{20}$$

into which one can inject a parameter λ if one exploits the symmetry $x \to x/\lambda$, $y \to \lambda y$ of Eq. (1). It is noted that the matrices in (20) are trace-free and hence belong to the Lie algebra $sl(4)$.

Remark (The dual Demoulin surface). On use of Eqs. (20), it is readily verified that the vector $\tilde{\boldsymbol{r}} = \sqrt{pq}\,\eta$ satisfies the equations

$$\tilde{\boldsymbol{r}}_{xx} = q\tilde{\boldsymbol{r}}_y + \frac{1}{2}\left(\frac{p_{xx}}{p} - \frac{1}{2}\left(\frac{p_x}{p}\right)^2 - q_y\right)\tilde{\boldsymbol{r}},$$

$$\tilde{\boldsymbol{r}}_{yy} = p\tilde{\boldsymbol{r}}_x + \frac{1}{2}\left(\frac{q_{yy}}{q} - \frac{1}{2}\left(\frac{q_y}{q}\right)^2 - p_x\right)\tilde{\boldsymbol{r}},$$

which can be obtained from (19) by a transformation $p \to q$, $q \to p$, being a discrete symmetry of (1). Hence $\tilde{\boldsymbol{r}}$ can be viewed as the position vector of a 'dual' Demoulin surface $\tilde{M}^2$. We recall that the point $\tilde{\boldsymbol{r}}$ is the intersection of the Wilczynski normal with the Lie quadric of the surface M^2 at the point $\boldsymbol{r}$. In the case of affine spheres ($p = q$), $\boldsymbol{r}$ and $\tilde{\boldsymbol{r}}$ satisfy the same equations so that the original and the dual surfaces are equivalent in the sense of projective geometry.

7. The 6×6 linear problem

As can be verified directly, system (1) implies the compatibility of the following second-order linear system for a pair of functions φ, ψ:

$$\varphi_{xx} = -\frac{q_x}{q}\varphi_x + p\psi_y, \tag{21a}$$

$$\psi_{xx} = -\frac{p_x}{p}\psi_x + q\varphi_y, \tag{21b}$$

$$\varphi_{xy} = -\frac{1}{q}\varphi, \tag{21c}$$

$$\psi_{xy} = -\frac{1}{p}\psi, \tag{21d}$$

$$\varphi_{yy} = -\frac{q_y}{q}\varphi_y + p\psi_x, \tag{21e}$$

$$\psi_{yy} = -\frac{p_y}{p}\psi_y + q\varphi_x. \tag{21f}$$

The latter reduces to the linear representation for the Tzitzeica equation in the case of $p = q$, $\varphi = \psi$. System (21a)–(21f) may be rewritten as the first-order 6×6 linear system

$$\begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^3 \\ \Psi^4 \\ \Psi^5 \\ \Psi^6 \end{pmatrix}_x = \begin{pmatrix} 0 & -\frac{1}{p} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{p} & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & \frac{q_x}{q} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ p & 0 & 0 & 0 & 0 & -\frac{q_x}{q} \end{pmatrix} \begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^3 \\ \Psi^4 \\ \Psi^5 \\ \Psi^6 \end{pmatrix},$$

$$\begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^3 \\ \Psi^4 \\ \Psi^5 \\ \Psi^6 \end{pmatrix}_y = \begin{pmatrix} -\frac{p_y}{p} & 0 & 0 & 0 & 0 & q \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & \frac{p_y}{p} & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{q} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{q} & 0 \end{pmatrix} \begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^3 \\ \Psi^4 \\ \Psi^5 \\ \Psi^6 \end{pmatrix}, \tag{22}$$

with $\varphi = \Psi^5$, $\psi = \Psi^2$. Note that a 'spectral' parameter λ may be inserted by means of the symmetry $x \to x/\lambda$, $y \to \lambda y$ of Eq. (1). It is also observed that the matrices in (22) are elements of the $so(3,3)$ Lie algebra which is isomorphic to $sl(4)$.

The appearance of the two Moutard equations (21c) and (21d) for φ and ψ, respectively, indicates the existence of a Bäcklund transformation based on the classical Moutard transformation. In order to explore the geometric content of such a Bäcklund transformation, we first have to explain the relationship between the linear systems (20) and (21a)–(21f) which is rooted in the classical Plücker correspondence between straight lines in P^3 and points in P^5.

8. The Plücker correspondence and the Godeaux sequence of a surface $M^2 \in P^3$

Let us consider a line l in P^3 passing through the points $\boldsymbol{a}$ and $\boldsymbol{b}$ with the homogeneous coordinates $\boldsymbol{a} = (a^0 : a^1 : a^2 : a^3)$ and $\boldsymbol{b} = (b^0 : b^1 : b^2 : b^3)$. With the line l we associate a point $\boldsymbol{a} \wedge \boldsymbol{b}$ in projective space P^5 with the homogeneous coordinates

$$\boldsymbol{a} \wedge \boldsymbol{b} = (p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}), \tag{23}$$

where

$$p_{ij} = \det \begin{pmatrix} a^i & a^j \\ b^i & b^j \end{pmatrix}. \tag{24}$$

The coordinates p_{ij} satisfy the well-known quadratic Plücker relation

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0. \tag{25}$$

Instead of $\boldsymbol{a}$ and $\boldsymbol{b}$ we may consider arbitrary linear combinations thereof without changing $\boldsymbol{a} \wedge \boldsymbol{b}$ as a point in P^5. Hence, the map (23)–(24) constitutes a well-defined Plücker correspondence $l(\boldsymbol{a}, \boldsymbol{b}) \mapsto \boldsymbol{a} \wedge \boldsymbol{b}$ between lines in P^3 and points on the Plücker quadric in P^5. If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are points in P^3 and κ is a scalar, the following properties hold:

$$\boldsymbol{a} \wedge \boldsymbol{b} = -\boldsymbol{b} \wedge \boldsymbol{a}, \quad \boldsymbol{a} \wedge \boldsymbol{a} = 0 \quad \text{(SKEW-SYMMETRY)},$$
$$\kappa\,(\boldsymbol{a} \wedge \boldsymbol{b}) = (\kappa\,\boldsymbol{a}) \wedge \boldsymbol{b} = \boldsymbol{a} \wedge (\kappa\,\boldsymbol{b}) \quad \text{(ASSOCIATIVITY)},$$
$$(\boldsymbol{a} + \boldsymbol{c}) \wedge \boldsymbol{b} = \boldsymbol{a} \wedge \boldsymbol{b} + \boldsymbol{c} \wedge \boldsymbol{b} \quad \text{(DISTRIBUTIVITY)},$$
$$(\boldsymbol{a} \wedge \boldsymbol{b})' = \boldsymbol{a}' \wedge \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b}' \quad \text{(LEIBNIZ RULE)}.$$

The Plücker correspondence plays an important role in the projective differential geometry of surfaces and often sheds some new light on those properties of surfaces which are not 'visible' in P^3 but acquire a precise geometric meaning only in P^5. Thus, let us consider a surface $M^2 \in P^3$ with position vector $\boldsymbol{r} = (r^0 : r^1 : r^2 : r^3)$ satisfying Eq. (5):

$$\boldsymbol{r}_{xx} = p\boldsymbol{r}_y + \pi\boldsymbol{r}, \qquad \boldsymbol{r}_{yy} = q\boldsymbol{r}_x + \chi\boldsymbol{r}.$$

Since two pairs of points $(\boldsymbol{r}, \boldsymbol{r}_x)$ and $(\boldsymbol{r}, \boldsymbol{r}_y)$ generate two lines in P^3 which are tangential to the x- and y-asymptotic lines, respectively, the formulae

$$U = \boldsymbol{r}_x \wedge \boldsymbol{r}, \quad V = \boldsymbol{r}_y \wedge \boldsymbol{r}$$

define the images of these lines under the Plücker embedding. Hence, with any surface $M^2 \in P^3$ there are canonically associated two surfaces $U(x, y)$ and $V(x, y)$ in P^5 lying on the Plücker quadric (25). In view of the formulae

$$U_x = pV, \quad V_y = qU,$$

we conclude that the line in P^5 passing through a pair of points (U, V) can also be generated by the pair of points (U, U_x) (and hence is tangential to the x-coordinate line on the surface U) or by a pair of points (V, V_y) (and hence is tangential to the y-coordinate line on the surface V). Consequently, the surfaces U and V are two focal surfaces of the congruence of straight lines (U, V) or, equivalently, V is the Laplace transform of U with respect to x and U is the Laplace transform of V with respect to y. We emphasize that the x- and y-coordinate lines on the surfaces U and V are not asymptotic but conjugate. Continuation of the Laplace sequence in both directions, that is taking the x-transform of V, the y-transform of U, etc., leads, in the generic case, to an infinite Laplace sequence in P^5 known as the Godeaux sequence of a surface M^2 [4, p. 344]. The surfaces of the Godeaux sequence carry important geometric information about the surface M^2 itself.

The case of a closed, i.e. periodic Godeaux sequence is particularly interesting. It turns out, that the only surfaces $M^2 \in P^3$ for which the Godeaux sequence is of period 6 (the value 6 turns out to be the least possible) are the surfaces of Demoulin [4, p. 360]. This result may be regarded as an equivalent geometric description of Demoulin surfaces.

In modern terminology, the classical Laplace sequence of conjugate nets [6] is governed by the two-dimensional Toda lattice [17]

$$(\ln h_n)_{xy} = -h_{n-1} + 2h_n - h_{n+1}.$$

There exists a Toda chain associated with any of the Kac–Moody Lie algebras [1]. For instance, the affine Lie algebra $A_5^{(1)}$ generates a Toda lattice of period 6:

$$\begin{aligned} (\ln h_1)_{xy} &= -h_0 + 2h_1 - h_2, & (\ln h_2)_{xy} &= -h_1 + 2h_2 - h_3, \\ (\ln h_3)_{xy} &= -h_2 + 2h_3 - h_4, & (\ln h_4)_{xy} &= -h_3 + 2h_4 - h_5, \\ (\ln h_5)_{xy} &= -h_4 + 2h_5 - h_6, & (\ln h_6)_{xy} &= -h_5 + 2h_6 - h_1. \end{aligned} \tag{26}$$

Hence, the above characterization of Demoulin surfaces suggests that Eq. (1) can be embedded in (27). This is indeed the case. The appropriate reduction corresponds to the $D_3^{(2)}$ subalgebra and is given by $h_6 = h_1, h_5 = h_2, h_4 = h_3$. System (27) now specializes to

$$(\ln h_1)_{xy} = -h_2 + h_1, \quad (\ln h_2)_{xy} = -h_1 + 2h_2 - h_3, \quad (\ln h_3)_{xy} = -h_2 + h_3.$$

It is readily verified that $(\ln h_1h_2h_3)_{xy} = 0$ so that $h_2 = 1/h_1h_3$ without loss of generality. Hence, we end up with the system

$$(\ln h_1)_{xy} = h_1 - \frac{1}{h_1h_3}, \qquad (\ln h_3)_{xy} = h_3 - \frac{1}{h_1h_3},$$

which coincides with (1) if we make the identifications $p = -1/h_1$, $q = -1/h_2$. We note that the linear system (21a)–(21f) is equivalent to the linear representation of the $D_3^{(2)}$ Toda lattice [20].

The Plücker construction also gives the relationship between the linear systems (20) and (21a)–(21f). A straightforward calculation reveals that (20) transforms into (21a)–(21f) if one sets

$$\varphi = \tfrac{1}{2}(\boldsymbol{r}_1 \wedge \boldsymbol{r}_2 + \boldsymbol{r} \wedge \boldsymbol{\eta}), \qquad \psi = \tfrac{1}{2}(\boldsymbol{r}_2 \wedge \boldsymbol{r}_1 + \boldsymbol{r} \wedge \boldsymbol{\eta}). \tag{27}$$

Here, it is convenient to be aware of the formulae

$$\varphi_x = \boldsymbol{r}_1 \wedge \boldsymbol{\eta}, \tag{28a}$$

$$\varphi_y = \frac{1}{2q}\boldsymbol{r}_2 \wedge \boldsymbol{r}, \tag{28b}$$

$$\psi_y = \boldsymbol{r}_2 \wedge \boldsymbol{\eta}, \tag{28c}$$

$$\psi_x = \frac{1}{2p}\boldsymbol{r}_1 \wedge \boldsymbol{r}. \tag{28d}$$

Insertion of the expressions for $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{\eta}$ as given by (13) into (27) and (28b) (28d) produces

$$\begin{aligned} \varphi &= -\frac{q}{2}\left(\frac{\boldsymbol{r}_y \wedge \boldsymbol{r}}{q}\right)_x, & \varphi_y &= \frac{1}{2q}\boldsymbol{r}_y \wedge \boldsymbol{r}, \\ \psi &= -\frac{p}{2}\left(\frac{\boldsymbol{r}_x \wedge \boldsymbol{r}}{p}\right)_y, & \psi_x &= \frac{1}{2p}\boldsymbol{r}_x \wedge \boldsymbol{r}. \end{aligned}$$

If we recall that $\boldsymbol{r}_x \wedge \boldsymbol{r} = U$, $\boldsymbol{r}_y \wedge \boldsymbol{r} = V$, we can bring these equations into the form

$$\varphi = -\frac{q}{2}\left(\frac{V}{q}\right)_x, \quad \psi = -\frac{p}{2}\left(\frac{U}{p}\right)_y, \quad \varphi_y = \frac{1}{2q}V, \quad \psi_x = \frac{1}{2p}U.$$

Geometrically, these formulae imply that φ is the Laplace transform of V in x-direction, while ψ is the Laplace transform of U in y-direction. Thus, our choice of variables is quite natural: both φ and ψ belong to the Godeaux sequence of the Demoulin surface M^2. We emphasize that, unlike U and V, φ and ψ do not lie on the Plücker quadric.

9. A Bäcklund transformation acting on Demoulin surfaces

In order to derive a Bäcklund transformation for Demoulin surfaces, it turns out convenient to consider the linear representation of the Demoulin system

$$\begin{aligned} (\ln h)_{xy} &= h - \frac{1}{hk}, \quad h = -\frac{1}{q}, \\ (\ln k)_{xy} &= k - \frac{1}{hk}, \quad k = -\frac{1}{p}, \end{aligned} \tag{29}$$

in the form

$$\varphi_{xx} = \frac{h_x}{h}\varphi_x + \lambda\frac{1}{k}\psi_y, \tag{30a}$$
$$\psi_{xx} = \frac{k_x}{k}\psi_x + \lambda\frac{1}{h}\varphi_y, \tag{30b}$$
$$\varphi_{xy} = h\varphi, \tag{30c}$$
$$\psi_{xy} = k\psi, \tag{30d}$$
$$\varphi_{yy} = \frac{h_y}{h}\varphi_y + \frac{1}{\lambda}\frac{1}{k}\psi_x, \tag{30e}$$
$$\psi_{yy} = \frac{k_y}{k}\psi_y + \frac{1}{\lambda}\frac{1}{h}\varphi_x. \tag{30f}$$

The linear system (21a)–(21f) is retrieved in the case $\lambda = -1$. We focus on the Moutard equations (30c) and (30d) and state the classical Moutard transformation [19]:

Theorem 1. *The Moutard equations*

$$\varphi_{xy} = h\varphi, \quad \psi_{xy} = k\psi \tag{31}$$

are form-invariant under the transformation

$$\begin{aligned} \varphi \to \varphi' &= \frac{S}{\varphi^0}, \qquad h \to h' = h - 2(\ln \varphi^0)_{xy}, \\ \psi \to \psi' &= \frac{T}{\psi^0}, \qquad k \to k' = k - 2(\ln \psi^0)_{xy}, \end{aligned} \tag{32}$$

where φ^0, ψ^0 is another scalar pair of solutions of (31) *and the bilinear potentials S, T are defined by the relations*

$$\begin{aligned} S_x &= \varphi^0\varphi_x - \varphi^0_x\varphi, & T_x &= \psi^0\psi_x - \psi^0_x\psi, \\ S_y &= \varphi^0_y\varphi - \varphi^0\varphi_y, & T_y &= \psi^0_y\psi - \psi^0\psi_y, \end{aligned} \tag{33}$$

which are compatible modulo (31).

In general, the structure of the remaining equations of (30a)–(30f) is not preserved by the Moutard transformation. However, the latter may be specialized according to the following:

Lemma 1. *If φ, ψ and φ^0, ψ^0 are solutions of the linear system* (30a)–(30f) *with parameters λ and μ respectively, then the integration constants in S and T may be chosen in such a way that*

$$\begin{aligned} S &= \alpha\varphi^0\varphi + \beta\psi^0\psi + \gamma\frac{\varphi^0_x}{h}\varphi_y + \delta\frac{\varphi^0_y}{h}\varphi_x + \rho\frac{\psi^0_x}{k}\psi_y + \sigma\frac{\psi^0_y}{k}\psi_x, \\ T &= \beta\varphi^0\varphi + \alpha\psi^0\psi + \rho\frac{\varphi^0_x}{h}\varphi_y + \sigma\frac{\varphi^0_y}{h}\varphi_x + \gamma\frac{\psi^0_x}{k}\psi_y + \delta\frac{\psi^0_y}{k}\psi_x, \end{aligned}$$

where the constants α, β, γ, δ, ρ and σ are given by

$$\alpha = \frac{\lambda^2+\mu^2}{\lambda^2-\mu^2}, \quad \rho = \sigma = -\beta = \frac{2\lambda\mu}{\lambda^2-\mu^2}, \quad \gamma = -\frac{2\lambda^2}{\lambda^2-\mu^2}, \quad \delta = -\frac{2\mu^2}{\lambda^2-\mu^2}.$$

Furthermore, the constraint

$$(\varphi^0)^2 - 2\frac{\varphi^0_x\varphi^0_y}{h} = (\psi^0)^2 - 2\frac{\psi^0_x\psi^0_y}{k} \tag{34}$$

is admissible.

It may be directly verified that S and T as given in the above lemma indeed satisfy the defining relations (33). The difference of the left-hand and right-hand sides of the constraint (34) constitutes a first integral of the linear system (30a)–(30f) which reflects the fact that the norm of the vector $(\Psi^1, \Psi^2, \Psi^3, \Psi^4, \Psi^5, \Psi^6)^{\mathrm{T}}$ in (22) is constant with respect to the Killing–Cartan metric of $so(3,3)$. Lemma 1 now puts us in a position to formulate the pivotal result of this section.

Theorem 2. *The Demoulin system* (29) *and its linear representation* (30a)–(30f) *are invariant under the Moutard-type transformation* (32) *with the specifications given in Lemma* 1.

The action of the above Bäcklund transformation on the surface M^2 may be derived in a purely algebraic manner. Indeed, the primed version of the Plücker embedding (27) and (28a)–(28d) constitutes quadratic equations for the Wilczynski tetrahedral $\boldsymbol{r}'$, $\boldsymbol{r}'_1$, $\boldsymbol{r}'_2$, $\boldsymbol{\eta}'$ which, up to a sign, possess a unique solution. A tedious but straightforward calculation results in the following *linear* action of the Bäcklund transformation:

Theorem 3. *If $\boldsymbol{r}$ is the position vector of a Demoulin surface M^2 then another Demoulin surface $M^{2'}$ is given by*

$$\boldsymbol{r}' = f^0\boldsymbol{r} + f^1\boldsymbol{r}_1 + f^2\boldsymbol{r}_2 \tag{35}$$

with coefficients

$$f^0 = -\frac{1}{2}\frac{\mu\varphi^0 + \psi^0}{\Gamma}, \quad f^1 = \mu\frac{\varphi_y^0}{h\Gamma}, \quad f^2 = \frac{\psi_x^0}{k\Gamma}$$

and

$$\Gamma = \sqrt{\left|(\varphi^0)^2 - 2\frac{\varphi_x^0\varphi_y^0}{h}\right|} = \sqrt{\left|(\psi^0)^2 - 2\frac{\psi_x^0\psi_y^0}{k}\right|}. \tag{36}$$

The surfaces M^2 and $M^{2'}$ are the focal surfaces of the W-congruence formed by the lines $(\boldsymbol{r}', \boldsymbol{r})$.

In the above theorem, $\boldsymbol{r}'$ has been scaled by an irrelevant constant factor. In order that the lines $(\boldsymbol{r}', \boldsymbol{r})$ form a W-congruence [10] two properties must hold. On the one hand, the asymptotic lines have to correspond on the surfaces M^2 and $M^{2'}$. By construction, this is indeed the case. On the other hand, by virtue of the definition of the Wilczynski tetrahedral, the new surface vector $\boldsymbol{r}'$ takes the form

$$\boldsymbol{r}' = g^0\boldsymbol{r} + g^1\boldsymbol{r}_x + g^2\boldsymbol{r}_y.$$

Thus, if we identify a surface in P^3 with a surface in Euclidean space with position vector $\boldsymbol{R} = (r^1/r^0, r^2/r^0, r^3/r^0)$, we obtain the transformation law

$$\boldsymbol{R}' = \boldsymbol{R} + \frac{g^1r^0\boldsymbol{R}_x + g^2r^0\boldsymbol{R}_y}{g^0r^0 + g^1r_x^0 + g^2r_y^0},$$

which implies that $\boldsymbol{R}' - \boldsymbol{R}$ is tangential to M^2. Moreover, since the Moutard transformation and its inverse have the same form, the line segment $\boldsymbol{R}' - \boldsymbol{R}$ is also tangential to the second surface $M^{2'}$. Consequently, the lines $(\boldsymbol{r}', \boldsymbol{r})$ form a W-congruence with M^2 and $M^{2'}$ being its focal surfaces.

10. 'One-soliton' Demoulin surfaces

The aim of this section is to construct the surfaces associated with the one-soliton solution of the Demoulin system (29). These may be generated by means of the Bäcklund transformation derived in the previous section acting on the surface associated with the trivial seed solution $h = k = 1$. However, it is instructive to digress for a moment and consider the bilinear form of the Demoulin system. Thus, if we set

$$h = \frac{c}{a}, \quad k = \frac{c}{b},$$

the Demoulin system may be brought into the form

$$\frac{c^2(\ln c)_{xy}+ab}{c^2}=\frac{a^2(\ln a)_{xy}+ac}{a^2},\qquad \frac{c^2(\ln c)_{xy}+ab}{c^2}=\frac{b^2(\ln b)_{xy}+bc}{b^2}.$$

Since one of the functions a, b or c is arbitrary, the above system may be separated into the three bilinear equations

$$D_xD_ya\cdot a=2(a^2-ac),\quad D_xD_yb\cdot b=2(b^2-bc),\quad D_xD_yc\cdot c=2(c^2-ab),\tag{37}$$

where D_xD_y is a Hirota bilinear operator [14], that is $D_xD_yf\cdot f=2(ff_{xy}-f_xf_y)$. The simplest Hirota ansatz

$$a=1+\alpha_1\mathrm{e}^{\eta}+\alpha_2\mathrm{e}^{2\eta},\quad b=1+\beta_1\mathrm{e}^{\eta}+\beta_2\mathrm{e}^{2\eta},\quad c=1+\gamma_1\mathrm{e}^{\eta}+\gamma_2\mathrm{e}^{2\eta},$$

$$\eta=\delta_1x+\delta_2y,$$

produces a set of algebraic equations for the constant coefficients $\alpha_1,\dots,\delta_2$ which turns out to admit two solutions. The first solution leads to $h=k$ and corresponds to the one-soliton solution of the Tzitzeica equation. The associated affine spheres have been generated and displayed in [21,23].

The second solution ($h\neq k$) is given by

$$a=4\mathrm{e}^{2\alpha}\cosh^2\alpha,\quad b=4\mathrm{e}^{2\alpha}\sinh^2\alpha,\quad c=2\mathrm{e}^{2\alpha}\cosh(2\alpha),$$

where $\alpha=(\kappa x+y/\kappa)/2$, $\kappa=$ const. without loss of generality. Hence, the solution of the Demoulin system (29) reads

$$h=1-\frac{1}{2}\frac{1}{\cosh^2\alpha},\quad k=1+\frac{1}{2}\frac{1}{\sinh^2\alpha}.\tag{38}$$

It represents two travelling waves (a hump and a trough) propagating at the same speed. Both h and k are nonzero but k diverges at $\alpha=0$. This implies that $p=-1/h$ and $q=-1/k$ are nonsingular. They are displayed in Fig. 1. It is emphasized that even though q vanishes at $\alpha=0$, the coefficients in the 'Gauss–Weingarten' equations (19) are nonsingular.

The above one-soliton solution is readily generated by means of the Bäcklund transformation (32). If we start with the seed solution $h=k=1$, a particular solution of the linear system (30a)–(30f) with parameter $\mu=\kappa^3$ is given by

$$\varphi^0=\mathrm{e}^{\mathrm{i}\beta}\cosh\alpha,\quad \psi^0=-\mathrm{e}^{\mathrm{i}\beta}\sinh\alpha,$$

where $\beta=\sqrt{3}(\kappa x-y/\kappa)/2$. In this case, the constraint (34) is identically satisfied with

$$\Gamma\sim\mathrm{e}^{\mathrm{i}\beta}\Delta,\quad \Delta=\sqrt{\cosh(2\alpha)},$$

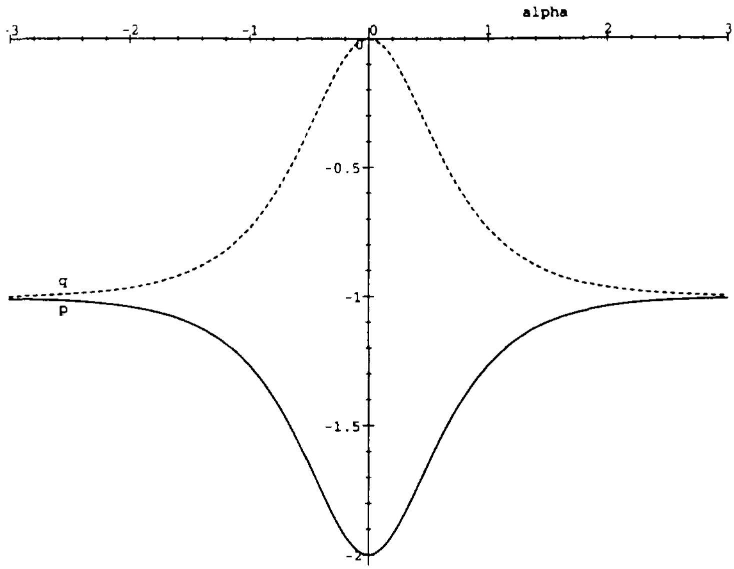

Fig. 1. The Demoulin one-soliton solution.

and h', k' coincide with the one-soliton solution (38). Up to a linear transformation, the seed surface vector reads

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ \mathrm{e}^{-(x+y)} \\ \mathrm{e}^{(x+y)/2}\cos\gamma \\ \mathrm{e}^{(x+y)/2}\sin\gamma \end{pmatrix}, \quad \gamma = \frac{\sqrt{3}}{2}(x-y)$$

so that the Bäcklund transformation (35) delivers the new real position vector (modulo a linear transformation)

$$\boldsymbol{r}' = \frac{1}{\Delta}\begin{pmatrix} \kappa^3\cosh\alpha - \sinh\alpha \\ \mathrm{e}^{-(x+y)}(\kappa\cosh\alpha + \sinh\alpha) \\ \mathrm{e}^{(x+y)/2}[(\kappa\cosh\alpha + \sinh\alpha)\sin\gamma + \sqrt{3}(\kappa\cosh\alpha - \sinh\alpha)\cos\gamma] \\ \mathrm{e}^{(x+y)/2}[(\kappa\cosh\alpha + \sinh\alpha)\cos\gamma - \sqrt{3}(\kappa\cosh\alpha - \sinh\alpha)\sin\gamma] \end{pmatrix}.$$

The case of the 'stationary' one-soliton solution, that is $\kappa = 1$, is of particular interest since in Euclidean or affine geometry, stationary one-soliton surfaces tend to be surfaces of revolution. Examples include the pseudo-sphere [9] and the simplest Tzitzeica surface of revolution [23]. In the present context, one may show that the only Demoulin surfaces of revolution are affine spheres. These have been discussed by Jonas in [15]. A typical affine sphere of revolution is shown in Fig. 2.

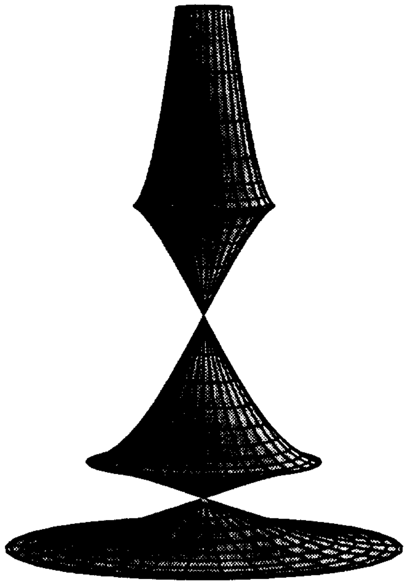

Fig. 2. An affine sphere of revolution.

It turns out that for $\kappa = 1$ the solutions r'^0 and r'^1 coincide. In order to obtain a fourth linearly independent solution we replace r'^1 by

$$\lim_{\kappa\to 1} \frac{r'^1 - r'^0}{\kappa - 1}.$$

Thus, l'Hospital's rule delivers the position vector

$$r' = \frac{1}{\Delta}\begin{pmatrix} e^{-\alpha} \\ e^{-\alpha}\left[\dfrac{2}{\sqrt{3}}\beta + (e^{-\alpha} - 3e^{\alpha})\cosh\alpha\right] \\ e^{2\alpha}\sin\beta + \sqrt{3}\cos\beta \\ e^{2\alpha}\cos\beta - \sqrt{3}\sin\beta \end{pmatrix},$$

which implies that the surface vector in Euclidean space is given by

$$\boldsymbol{R}' = \begin{pmatrix} \dfrac{2}{\sqrt{3}}\beta + (e^{-\alpha} - 3e^{\alpha})\cosh\alpha \\ e^{3\alpha}\sin\beta + \sqrt{3}e^{\alpha}\cos\beta \\ e^{3\alpha}\cos\beta - \sqrt{3}e^{\alpha}\sin\beta \end{pmatrix}. \tag{39}$$

The parametrization (39) shows that the surface is generated by the curve $\boldsymbol{R} = \boldsymbol{R}(\alpha, \beta = 0)$ which is uniformly rotated and translated. This 'stationary one-soliton' Demoulin surface is depicted in Fig. 3 in terms of α and β.

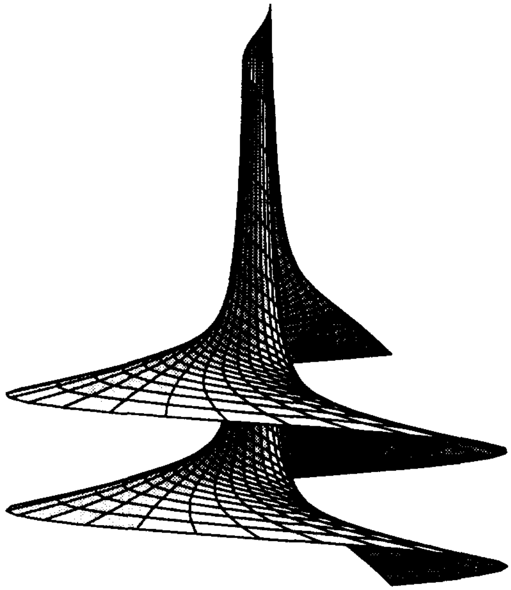

Fig. 3. A Demoulin one-soliton surface.

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